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V. *On the Variation in the Mean Motion of the Comet of Encke, produced by the resistance of an ether. By M.* Ottaviano Fabrizio Mossotti, *Uno dei XL della Società Italiana delle Scienze.* (Translated from the French by Dr. Gregory.)

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IN Vol. ix. No. 2. of the Astronomical Correspondence, published at Genoa by that learned promoter of the science of Astronomy, Baron Zach, a summary has appeared of the work of Mr. Encke on the comet, which he was the first to bring into particular notice, and which justly bears his name. This indefatigable calculator, after having in vain attempted to account for the diminution of the periodic time of the comet at each revolution by the perturbation of the planets, was led to conclude that some other cause must exist to disturb its motion. "The most natural of the causes which we can suppose," says Mr. Encke, "is the resistance of an ether diffused in space, which produces a diminution both in the periodical times and in the eccentricity." He goes on to give validity to this supposition, by demonstrating that the diminution which would result from it in both these elements would be in the same ratio as observations have shown us is actually the case. One difficulty presents itself, notwithstanding, against this supposition. If an ether exist, how is it that the planets have not yet displayed any effects as arising from it? Can it be supposed that, on account of the eccentric figure of the orbit, the perpetual changes in the form and volume, and the smallness of the mass of comets, the resistance of an ether would become sensible in so short a time with respect to them, and yet produce effects so trifling on the planets that they cannot even be discovered till after a lapse of much time and a series of future observations? I interrogated the Calculus for some information on this subject, and I found the reply in the affirmative.

When we think of calculating the resistance, which the motion of a comet may experience from an ether diffused in space, two great difficulties at first present themselves: the first is to find out the law of the density of the ether thus diffused in space; the second is to take account of the continual changes

in the figure and volume of the comet, as it approaches to, or recedes from, the Sun. On these two subjects we can only raise hypotheses; but if, with the assistance of some hypotheses in accord with our present knowledge, we can find a sensible secular variation in the motion of the comets, though not applicable, at least hitherto, to any planet,—we may hence conclude that the existence of such an ether is not disproved by the known fact that its influence has not yet sensibly affected the motions of the planets.

In order to direct our researches on this subject, we will adopt the formulæ of the second part of the *Mécanique Analytique*, section 7. § 3. retaining all the denominations employed by the celebrated author. Taking a for the semi-axis major, r for the radius vector, and $\sqrt{g}$ for the number 0·0172021, these formulæ give for the secular variation of the semi-axis major,

$$\delta a = -2 a^2 \Gamma \left(\frac{2}{r} - \frac{1}{a}\right)^{\frac{3}{2}} \sqrt{g}\, \delta t$$

and for the secular variation of the mean anomaly,

$$\delta u = \frac{1}{\sqrt{a^3}} \sqrt{g}\, \delta t$$

In these equations the quantity Γ ought to be considered as different in two different circumstances. The first of these circumstances obtains when the celestial bodies *do not* change in magnitude and figure; in this case the quantity Γ is proportional to the density of the medium; and the density may be supposed a function of the distance r of the celestial body from the Sun. The second case subsists, on the contrary, when the bodies, the motion of which is considered, *do* change continually in magnitude and figure; and in this case, the quantity Γ is in proportion to the density of the medium multiplied by a function depending on the varying figure and magnitude of the bodies.

Let us begin by examining the first case, viz. that of the planets, the volumes and figures of which may be considered as constant: and here, with regard to the density of the medium, we will adopt with Mr. Encke the hypothesis of Newton, who has supposed that this density diminishes in the inverse ratio of the square of the distance from the Sun. In this case we shall have

$$\Gamma = \frac{f}{r^2}$$

f being a fixed quantity, depending on the figure, volume and mass, of the planet, as well as on the density of the medium at the distance taken for unity.

By the substitution of this value of Γ in the formula of the variation of the semi-axis major, we deduce

$$\delta a = -2f\frac{a^2}{r^2}\left(\frac{2}{r}-\frac{1}{a}\right)^{\frac{3}{2}}\sqrt{g}\,\delta t$$

If it be remembered that the formulæ of the elliptical motion (see *Mec. Anal.* vol. ii. p. 18, 19) give

$$r = a\,(1 - e\cos\theta); \quad \text{and } \delta t = \sqrt{\frac{a^3}{g}}\,(1 - e\cos\theta)\,\delta\theta$$

by reducing the whole into a function of the eccentric anomaly θ, we shall have

$$\delta a = -2f\,\frac{(1+e\cos\theta)^4}{(1-e^2\cos^2\theta)^{2}}\times\frac{\delta\theta}{\sqrt{(1-e^2\cos^2\theta)}}$$

In the integration of the second member of this equation, we only need the non-periodical part of the integral, because this is the only part which can give secular variations. Now, by representing in general by $\nu^{(2n+1)}\,\theta$ the part proportional to θ in the integral of the function $\dfrac{1}{(1-e^2\cos^2\theta)^{\frac{2n+1}{2}}}$ we find that the non-periodical part of the integral of the preceding equation is reduced to

$$a = A - 2f\left\{\nu^{(1)} - 8\,\nu^{(3)} + 8\,\nu^{(5)}\right\}\theta$$

A being the value of a when $\theta = 0$.

The quantities $\nu^{(1)}$, $\nu^{(2)}$, $\nu^{(3)}$, can all be calculated by means of the semi-circumference of the circle, and of the complete elliptical functions F^1 and E^1 of the first and second kind, corresponding to the eccentricity e. In short, by making $b^2 = 1 - e^2$ it is easy to deduce

$$\nu^{(-1)} = \frac{2}{\pi}E^1$$

$$\nu^{(1)} = \frac{2}{\pi}F^1$$

$$\nu^{(3)} = \frac{1}{b^2}\nu^{(-1)}$$

$$\nu^{(5)} = \frac{2}{3}\left(1+\frac{1}{b^2}\right)\nu^{(3)} - \frac{1}{3b^2}\nu^{(1)}$$

so that all these quantities will be easily calculated with the assistance of M. LEGENDRE's tables of the functions E^1 and F^1.

The eccentric anomaly θ is given as a function of the time by a series, the first term of which is $t\sqrt{\frac{g}{A^3}}$, and all the other terms are periodical

functions. In considering only the secular variations, we may then substitute $t\sqrt{\frac{g}{A^3}}$ for θ in the superior equation; and by this substitution, making for abbreviation

$$2\left(\nu^{(1)} - 8\nu^{(3)} + 8\nu^{(5)}\right)\sqrt{\frac{g}{A^3}} = P$$

the value of the semi-axis major will be represented by the formula,

$$a = A - fPt$$

Let us substitute this expression of the semi-axis major in the equation of the mean anomaly, and perform the integration relatively to t. By causing the value of u to commence with $t = 0$, we shall have

$$u = \frac{2\sqrt{g}}{fP}\left\{\frac{1}{(A - fPt)^{\frac{1}{2}}} - \frac{1}{A^{\frac{1}{2}}}\right\}$$

or by developing and retaining only the terms below the second power of f,

$$u = t\sqrt{\frac{g}{A^3}}\left(1 + \frac{3fPt}{4A}\right)$$

an equation which will give for any time the mean anomaly corrected by that part of the secular variation arising from the resistance of the medium.

The planet, which, on the hypothesis adopted for the law of the density of the medium, ought to experience the greatest resistance, is Mercury, on account of his proximity to the Sun, and his superior velocity. For this planet, supposing the eccentricity of his orbit equal to 0·2055 of the semi-axis major, and the length of the semi-axis major equal to 0·3871 of the mean distance of the earth from the Sun, we find by the preceding formula,

$$a = 0{\cdot}3871 - 0{\cdot}17052ft$$
$$u = 14733''t\,(1 + 0{\cdot}33038ft)$$

These values, compared with those analogous to them, and which we will find for the comet, and corresponding to the same value of f, will make known to us the relations of the secular variations of the semi-axis major, and of the mean motions of the two heavenly bodies.

Let us now, therefore, pass on to consider the case of comets, whose figures and volumes change, according to their distance, from the Sun. All that observations have taught us in this respect, is, that when comets approach their perihelion, they become surrounded with an atmosphere which considerably augments their volume, and that this augmentation proceeds with great rapidity. The co-efficient f, which, as before observed, depends on

the figure and volume of the comet, and was constant, will consequently become rapidly variable in the approach of the comet to the perihelion To represent this circumstance by an algebraic formula, we will substitute the formula $f\left(\alpha + \frac{\beta}{r^2} + \frac{\gamma}{r^4}\right)$ for the quantity f, supposing that the quantities, α, β, γ, be so taken that the value of the formula shall change rapidly for the variations in the values of r, which are *below* unity, and slowly for those *above*. According to this hypothesis we shall have

$$\Gamma = \frac{f}{r^2}\left(\alpha + \frac{\beta}{r^2} + \frac{\gamma}{r^4}\right)$$

and by substituting this expression for Γ in the formula of the variation of the semi-axis major, it will produce

$$\delta a = -2f\frac{a^2}{r^2}\left(\alpha + \frac{\beta}{r^2} + \frac{\gamma}{r^4}\right) \times \left(\frac{2}{r} - \frac{1}{a}\right)^{\frac{3}{2}} \times \sqrt{g}\,\delta t$$

or, by introducing the eccentric anomaly,

$$\delta a = -2f\left(\alpha + \frac{\beta}{a^2} \cdot \frac{(1 + e\cos\theta)^6}{(1 - e^2\cos^2\theta)^2} + \frac{\gamma}{a^4} \cdot \frac{(1 + e\cos\theta)^8}{(1 - e^2\cos^2\theta)^4}\right) \cdot \frac{\delta\theta}{(1 - e^2\cos^2\theta)^{\frac{1}{2}}}$$

The part proportional to θ in the integral of this equation can be calculated equally by the complete elliptic functions F^1 and E^1. In short, by first performing the integration, and retaining the denominations adopted, we find

$$a = A - 2f\left\{ \alpha\nu^{(1)} - \left(8\alpha + \frac{\beta}{a^2}\right)\nu^{(3)} + \left(8\alpha + 18\frac{\beta}{a^2} + \frac{\gamma}{a^4}\right)\nu^{(5)} \right.$$
$$\left. - \left(48\frac{\beta}{a^2} + 32\frac{\gamma}{a^4}\right)\nu^{(7)} + \left(32\frac{\beta}{a^2} + 160\frac{\gamma}{a^4}\right)\nu^{(9)} - 256\frac{\gamma}{a^4}\nu^{(11)} + 128\frac{\gamma}{a^4}\nu^{(13)} \right\}\theta$$

We have then, in order to calculate successively the quantities ν, the equation

$$\nu^{(2n+3)} = \frac{2n}{2n+1}\left(1 + \frac{1}{b^2}\right)\nu^{(2n+1)} - \frac{2n-1}{2n+1} \cdot \frac{1}{b^2}\nu^{(2n-1)}$$

so that all these quantities will be reduced to functions of $\nu^{(-1)}$ and $\nu^{(1)}$, or, as we have before observed, of F^1 and E^1.

By changing, in the preceding formula, θ into $t\sqrt{\frac{g}{A^3}}$, and by supposing P equal to

$$-2\left\{ \alpha\nu^{(1)} - \left(8\alpha + \frac{\beta}{a^2}\right)\nu^{(3)} + \left(8\alpha + 18\frac{\beta}{a^2} + \frac{\gamma}{a^4}\right)\nu^{(5)} - \left(48\frac{\beta}{a^2} + 32\frac{\gamma}{a^4}\right)\nu^{(7)} \right.$$
$$\left. + \left(32\frac{\beta}{a^2} + 160\frac{\gamma}{a^4}\right)\nu^{(9)} - 256\frac{\gamma}{a^4}\nu^{(11)} + 128\frac{\gamma}{a^4}\nu^{(13)} \right\}\sqrt{\frac{g}{A^3}};$$

we shall have for the expression of the semi-axis major, with its secular variation, $a = A - fPt$: and, by the same procedure which gave us

the mean anomaly in the preceding case, we shall have likewise the mean anomaly

$$u = t\sqrt{\frac{g}{A^3}}\left(1 + \frac{3fP}{4A}\right)$$

With a view to apply these formulæ to the motion of the comet of Encke, we will adopt the following numbers: viz.

$$\alpha = 6, \quad \beta = -120, \quad \gamma = 1024,$$

which appear to us to answer satisfactorily to the above-mentioned conditions, and to the appearances which comets present to us. I would observe in the first place, that we may believe that comets have in general a very small mass, because their action on the motion of the planets is not sensible even at very small distances. Let us then suppose that the mass of the comet is $\frac{1}{100}$ of that of Mercury; and let us admit, as it is generally adopted, that the co-efficient of $\frac{1}{r^3}$ in the expression of Γ, belonging both to the planet and the comet, is in the direct ratio of the square of the diameter of the impingeing body, and also in the inverse ratio of the mass. After these hypotheses, by comparing the two expressions of Γ, and calling d the diameter of Mercury and d' that of the comet, we have

$$d' = \frac{d}{10}\left\{\alpha + \frac{\beta}{r^3} + \frac{\gamma}{r^4}\right\}^{\frac{1}{2}} *$$

If now we take $6''\cdot 6$ for the value of the diameter of Mercury seen at the mean distance of the earth from the Sun, and calculate by the assistance of this formula what ought to be the corresponding magnitudes of the diameters of the comet in the different positions of its orbit, we shall find that they go on augmenting in the following progression:

at aphelion	$r = 4\cdot 09$	app. diameter	=	0′ 1″·0
	$r = 3$	app. diam.	=	0 1 ·3
	$r = 2$	app. diam.	=	0 4 ·2
	$r = 1$	app. diam.	=	0 19 ·9
at perihelion	$r = 0\cdot 332$	app. diam.	=	3 10 ·3

Hence we see that our formula, though entirely hypothetical, gives a progression to the augmentation of the volume of the comet, agreeable to that which observations have made evident to us.

* This formula is deduced on the supposition that the figure of the comet is spherical, which is not the case; but as it is only used to show the progressive increase in the volume of the comet, on its approaching the Sun, by the assigned value of α, β, γ, this supposition will afford a sufficient idea.

After this reflection, let us introduce the supposed values of α, β, γ, into the expression of P, and let us adopt (as Mr. ENCKE has nearly ascertained them) the number, of which the tabular logarithm is 0·345 for the value of the semi-axis major, and the number 0·85 for the ratio of the eccentricity to the semi-axis major. With these numbers given, we find the numerical value of $P = 17920$, and consequently

$$a = A - 17920ft$$
$$u = 1076''t\,(1 + 6073ft)$$

The first difference between two consecutive terms, $t + 1$ and t, of this last expression of the mean anomaly diminished by 1076″, gives the secular diurnal variation of this anomaly after the time t. By comparing the expression of this variation after a revolution, that is after 1204 days, with the variation which Mr. ENCKE has deduced from observations, and which is 0″·1199 daily, we shall have a means of determining the co-efficient f. Then the value of this co-efficient, transferred to the equation of the mean anomaly of Mercury, will show us if any difference, which observations may have made known to us, has resulted in the mean anomaly of this planet.

By following this procedure, we shall first have

$$2''\cdot 1076\ .\ 1204{,}5\ .\ 6073f = 0''\cdot 1199$$

whence we deduce $f = 0\cdot 000{,}000{,}000{,}007{,}616{,}8$

This value of f, substituted in the equation of the mean anomaly of Mercury, which we have before found, gives

$$u = 14733''t\,(1 + 0\cdot 000{,}000{,}000{,}002{,}515{,}8t)$$

In 100 Julian years, or 36525 days, this formula gives for the difference of the mean anomaly the small quantity of 49″·5, which would produce a difference of no more than 31″·2 in the mean geocentric longitude of this planet in its inferior conjunctions; a difference the existence of which we cannot verify at so distant a period *.

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* If we calculate what ought to be the density of the ether, corresponding with the value already found of f by the formula

$$f = \frac{6\,n\,.\,\pi\,d^2}{4\,m\,.\,\pi\,d^3} = \frac{6\,n}{4\,m\,d},\ \text{where } d = 6''\cdot 6$$

denotes the diameter of Mercury, m his density, and n the density of the ether; and suppose the density of Mercury 10397 times greater than the density of the atmospheric air, we shall find $n = 0\cdot 000{,}000{,}000{,}001{,}777$; that is, that the density of the ether, which would produce the acceleration observed in the mean motion of the comet, will be about 360000 million times less than that of the atmospheric air.

We may therefore conclude, that, on the hypotheses adopted, the comet may have experienced from an ether a resistance, such as is required to make the calculus accord with observations, though the planets have not yet manifested the least effect of the existence of this ether. As nothing opposes the probability that the hypotheses which we have made, or some others analogous to them, are really correct; and as, moreover, the effect of the acceleration of the mean motion of this comet supports the opinion of the existence of an ether, a greater degree of credit will no doubt be assigned to the hypotheses. If the comet, which we expect eleven years hence, display corresponding effects, we shall then be authorised to regard the diffusion of an ether in celestial space as an admissible fact; and comets will have had the honour of being the first to enlighten us on this interesting point in the structure of the heavens.